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Solve the Hanoi Puzzle Using Difference Equations

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Abstract. This research aims to solve an old riddle invented by the French mathematician (Edward Lucas) in the year 1883 AD, known as (Tower of Hanoi puzzle) using difference equations, which we will address in terms of its concept, some characteristics, types, methods of solving each type of them and some illustrative examples.

Key words and phrases. Difference Equations. Homogenous first order Linear Difference Equations.Higher order Linear Difference with constant coefficients.

INTRODUCTION

The Tower of Hanoi is an ancient mathematical puzzle that is said to be inspired by an ancient legend revolving around the existence of a temple in Hanoi in Vietnam. Inside it is a large room with three columns with sixty-four golden discs of different sizes arranged from largest to smallest in the first column. Temple priests believe that A new prophecy will come if all the disks in the first column are moved to the last column, but according to certain conditions

search problem:

Solve the puzzle according to its terms mathematically using the equations of differences through two methods, either moving the discs with the least number of movements or with the largest number of movements

Previous studies :

Hasker P.^Davis 25 May 2002.

Journal of Applied Analysis and Computation Website:http://jaac-online.com/

Volume 46, Issues 1–2, June–July 2001, Pages 95-99

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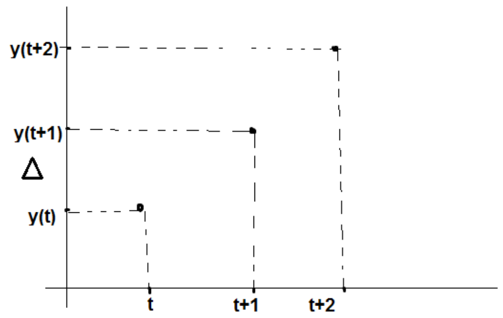
A longitudinal study of the performance of the elderly and young on the tower of hanoi puzzle and rey recall

In his research Normal performance on the Tower of Hanoi puzzle by amnesic patients has been taken as support for viewing this problem solving task as having a nondeclarative memory component. Individuals in each decade of life between the 20s and 80s were asked to solve this puzzle four times in four sessions with intersession intervals from 1 to 7 days.

1. Difference Equations .

In this chapter, we will get acquainted with the equations of differences, study some of their properties and types, and solve each of their types.

Definition 2.1.[1] *The difference equation is known as a regressive defined equation*



And write the difference equation for the function y(t) in the following form:

$$\Delta y(t) = y(t+1) - y(t)$$

Remark 2.2.[6] *The symbol Δ is called (Difference Operator)*

Example 2.3.[1] Find a value $\Delta y(t)$ if $y(t) = t^2$

Sol :

$$\Delta y(t) = y(t+1) - y(t)$$
$$\Delta y(t) = (t+1)^2 - t^2$$
$$= t^2 + 2t + 1 - t^2$$
$$= 2t + 1$$

Definition 2.3.[1] *If we have* $y(t,1)$ *then:*

$$1 - \quad \Delta_t y(t,x) = y(t+1,x) - y(t,x)$$
$$2 - \quad \Delta_x y(t,x) = y(t,x+1) - y(t,x)$$

where $\Delta_t y(t,x)$ *is the partial difference with respect to the operand* t*, and* $\Delta_x y(t,x)$ *is the partial difference with respect to the operand* x

Example 2.4.[2] Calculate the difference for the modulus x and for the modulus t if we have $y(t,x) = xt + t^2$

Sol:

$$\Delta_t y(t,x) = ((x+1)t + t^2) - (xt, t^2)$$
$$= xt + x + t^2 + 2\text{t} + 1 - \text{xt} - t^2$$

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$$= 2t + x + 1$$

$$\Delta_t y(t,x) = 2t + x + 1$$

Example 2.5.[1] Find a value $\Delta y(t)$ if $y(t) = 8^t$

Sol: $\Delta 8^t = (8-1)8^t = 7.8^t$

Theorem 2.6.[3] *Let us have two functions $y(t), z(t)$ where $t \in N$ and let K be constant, then:-*

$$1 - \quad \Delta\big(ky(t)\big) = k\Delta y(t)$$

Proof :

$$\Delta\big(ky(t)\big) = ky(t+1) - ky(t)$$

$$= k\Delta y(t)$$

$$\Delta\big(ky(t)\big) = k\Delta y(t)$$

$$2 - \quad \Delta\big(ky(t)\big) = k\Delta y(t)$$

Proof :

$$\Delta\big(y(t)+z(t)\big) = y(t+1) + z(t+1) - \big(y(t)+z(t)\big)$$

$$= y(t+1) - y(t) + z(t+1) - z(t)$$

$$= \Delta y(t) + \Delta z(t)$$

$$= \Delta\big(y(t)+z(t)\big) = \Delta y(t) + \Delta z(t)$$

Example 2.7.[4] let $y(t) = 2t$ *and* $z(t) = t + 1$*where* $t \in Z_+$ find $= \Delta\big(y(t) + z(t)\big)$

Sol:

$$\Delta\big(y(t)+z(t)\big) = \Delta y(t) + \Delta z(t)$$

$$\Delta\big((2t)+(t+1)\big) = \Delta(2t) + \Delta(t+1)$$

$$= \Delta(2(t+1) - 2t) + \big((t+2) - (t+1)\big) = 3$$

Definition 2.8.[1] *let $y(t)$ be a function then $\Delta^n y(t)$ is called (nth difference) It is defined as:*

$$\Delta^n y(t) = \Delta(\Delta^{n-1} y(t))$$

Definition 2.9.[2] *The order of the difference equation is defined as the highest order of difference of a function .*

Definition 2.10.[6] *The degree of the variance equation is known as the largest power of the variance of a function .*

Example 2.11.[6] let $(\Delta^{n-1} y(t))^{\,3} + \Delta y(t) - 3$ It is a second-order and third-order equation

Definition 2.12.[1] *The function $\sum y(t) = z(t) + c$ where t is a positive integer and c is a constant*

Theorem 2.13.[3] *Let $y(t), z(t)$ be two functions where $t \in Z_+$ and c is a constant , then:-*

1- $\sum cy(t) = c \sum y(t)$ *where c is a constant*

2- $\sum(y(t) + z(t)) = \sum y(t) + \sum z(t)$

Example 2.14.[1] find $\sum(y(t) + z(t))$ if $y(t) = 8t^5$, $z(t) = 3e^{-2t}$

Sol:

$$\sum(y(t) + z(t)) = \sum(8t^5 + 3e^{-2t})$$

$$= \frac{8}{6} t^5 + \frac{3}{e^{-2} - 1} (e^{-2})^t + c$$

Theorem 2.15.[3] *Let $z(t)$ be the indefinite summation of $y(t)$ then :*

$$\sum_{k=m}^{n-1} yk = [Zk]_m^n = Z_n - Z_m$$

Proof :

Suppose $Z(n)$ is the indefinite summation of the function $y(t)$ then:

$$\sum_{k=m}^{n-1} yk = y_m + y_{m+1} + \dots\dots\dots\dots + y_{n-1} =$$

$$\Delta Z_m + \Delta Z_{m+1} + \dots\dots\dots\dots + \Delta Z_{n-1} =$$

$$= (Z_{m+1} - Z_m) + (Z_{m+2} - Z_{m+1}) + \cdots \dots\dots (Z_n - Z_{n-1}) =$$

$$= Z_n - Z_{m=}$$

$$= [Zk]_m^n$$

2.1. Classify linear difference equations with constant coefficients [7]

Linear difference equations with fixed coefficients are classified into three types that we will discuss in this section with the methods of solving each of them

2.1.1. Homogenous First Order linear Difference Equations [7]

This type is written in the following format:

$$y(t+1) = p(t)y(t) = r(t) \quad \rightarrow \quad r(t) = 0$$

2.1.2. Homogenous First Order linear Difference Equations [8]

The general form of this type is as follows:

$$y(t+1) = p(t)y(t) = r(t), \quad t \in D_{y(t)} \;\rightarrow\; D_{y(t)} \text{ is the range of the function}$$

2.1.3. Homogenous First Order linear Difference Equations [8]

The general form of this type is as follows:

$$y(t+a) + a_{n-1}y(t+n-1) + \dots\dots\dots + a_0 y(t) = 0 \quad \rightarrow \quad a_0 \neq 0$$

3. Tower of Hanoi

By chance, the French mathematician Edouard Lux learned in 1883 that there is an ancient legend in the temple of Hanoi in Vietnam that predicted that there was a good prophecy that would come if he solved the puzzle in this temple, which is three columns found in the first column sixty-four discs of different sizes Arranged from largest to smallest, the scientist found that the puzzle can be solved mathematically according to conditions set by the temple

3.1. puzzle goal

Transfer all the discs from the first column to the third column with the help of the second column

3.2. puzzle terms

1-Transfer only one disk at a time.

2-Do not place the largest disc on top of the smaller one.

3-When moving a disk from one column to another, movement is calculated.

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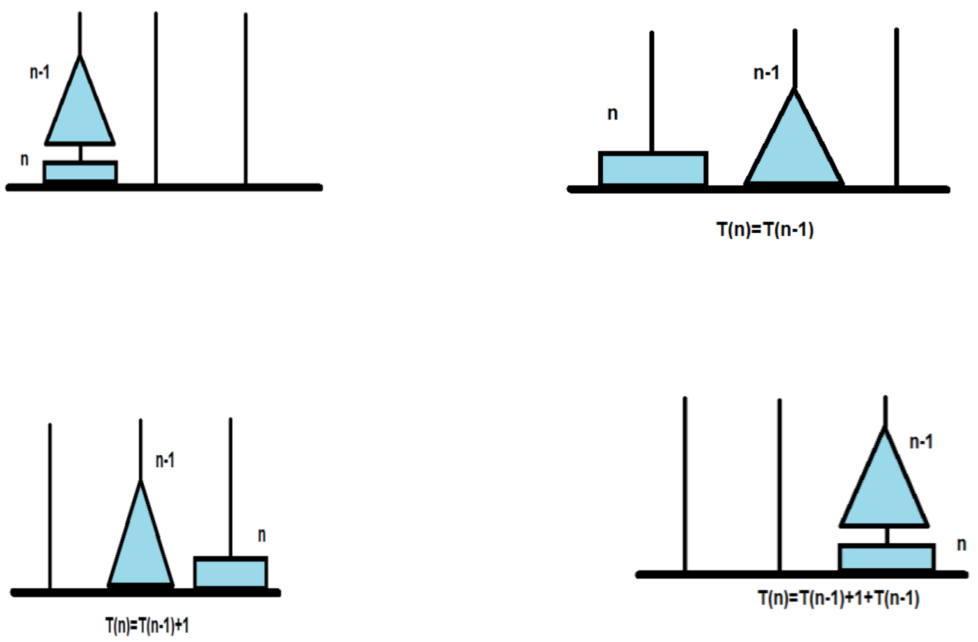
3.3. The first method is to transfer discs with the fewest number of movements

We will use difference equations to calculate the minimum number of moves to move the discs

Suppose n denotes the number of discs and $m(n)$ the minimum number of moves to move n of discs

The following figures show the stages of transferring n from the discs from the first column to the last with the least number of movements and how to derive the difference equation required to calculate $m(n)$



Therefore, we will have the following differences:

$$m(n) = 2m(n-1) + 1$$

Therefore

$$m(n) = 2m(n) + 1$$

$$\therefore\ m(n+1) - 2m(n) = 1 \qquad \rightarrow \quad (1)$$

Equation (1) is a linear and non-homogeneous difference equation in it $p(n)2, r(n) = 1$ with the initial condition $m(0) = 0$ and therefore its solution will be as follows:

$$m(n) = g(n) \sum \frac{r(n)}{Eg)n)} + C \qquad \rightarrow \quad (2)$$

So, let's be the following equation:

$$m(n+1) - 2m(n) = 0 \qquad , n \in N$$

It is the homogeneous equation of the equation (1) and can be resolved in the repetitive way

Now to assume that $g(n)$ is the solution to the homogeneous equation will be the following form

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$$g(n) = \prod_{k=1}^{n-1} p(n) = 2^{n-1}$$

$$\therefore g(n) = 2^n$$

Now let us calculate the equation (2) by compensation for each of $g(n), Eg(n), r(n)$, so we get:

$$m(n) = 2^n(\sum \frac{1}{2^{n+1}} + C)$$

$$= \sum \frac{1}{2^{n+1}} + 2^n.c$$

$$\left(\frac{2^n}{2}\right) - 2.\left(\frac{1}{2^n}\right) + 2^n.c$$

$$\therefore m(n) = -1 + 2^n.c$$

$$m(0) = 0$$

$$\therefore 0 = -1 + 2^0.c$$

$$\therefore c = 1$$

Now we conclude that the minimum number of movements to move n from the disks is

$$m(n) = 2^n - 1$$

The following table shows the least number of possible movements to move n from disks where $n = 1, \dots \dots .., 64$

table number (1)

n	$2^n + 1$
1	*1*
2	*3*
3	*7*
4	*15*
5	*31*
6	*63*
7	*127*
8	*255*
9	*511*
10	*1023*
. . .	. . .
64	*1.844674407 x* 10^{19}

3.4. The second method

We will also use the difference equation to calculate the largest number of movements to move the discs.

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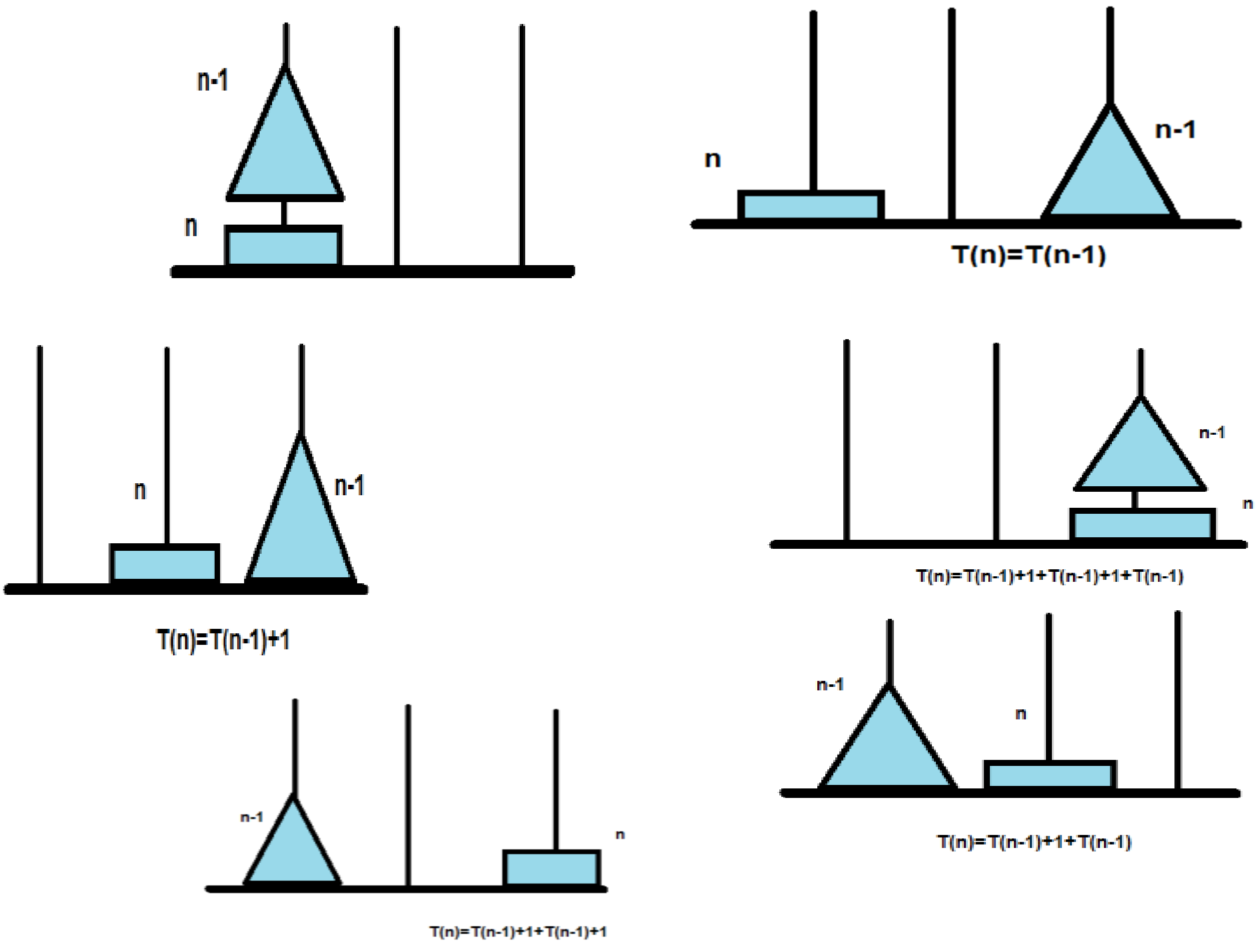
Suppose n represents the number of disks and $M(n)$ represents the largest number of moves to move n from disks

The following figures show the stages of transferring n from the discs from the first column to the last with the largest number of movements and how to derive the difference equation required to calculate $M(n)$.



So we will have the following equation of differences:

$$M(n) = 3M((n-1) + 2$$

$$M(n)(n+1) = 3M(n) + 2$$

$$:.\, M(n+1) - 3M(n) + 2 \qquad (3)$$

Equation (3) is a linear and inhomogeneous difference equation in it $r(n) = 2\,, p(n) = 3$ with the initial condition $M(0) = 0$

It will be solved as follows:

$$M(n) = G(n)\left(\sum \frac{R(n)}{EG(n)} + D\right) \qquad (4)$$

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So let's first have the following equation:

$$M(n+1) - 3M(n) = 0 \qquad ,n \in N$$

It is the homogeneous equation accompanying equation (3) and can be solved by iterative method.

Suppose that $G(n)$ is the solution to the homogeneous equation and it will be as follows:

$$G(n) = \prod_{k=1}^{n-1} p(n)$$

$$\therefore G(n) = 3^n$$

Now let's calculate equation (4) by substituting for both of $R(n), EG(n), G(n)$ and we get that :

$$M(n) = 3^n(\sum \frac{2}{3^{n+1}} + D)$$

$$= 3^n \sum \frac{2}{3^{n+1}} + 3^n.D$$

$$= 3^n.\left(\frac{2}{3}\right)\left(\frac{-3}{2}.\left(\frac{1}{3}\right)^n\right) + 3^n.D$$

$$M(n) = -1 + 3^n.D$$

$$M(0) = 0$$

$$\therefore D = 1$$

We conclude that the largest number of possible movements to move n from disks is:

$$M(n) = 3^n - 1$$

The following table shows the largest number of possible movements to move n from disks where $n = 1, \dots \dots, 64$

table number (2)

n	$3^n + 1$
1	*2*
2	*8*
3	*26*
4	*80*
5	*242*
6	*728*
7	*2186*
8	*6560*
9	*19682*
10	*59048*
.	.
.	.
.	.

64	$3.43368382 \times 10^{30}$

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